QUIZ 8 SOLUTION: LESSON 10
SEPTEMBER 20, 2017

Write legibly, clearly indicate the question you are answering, and put a box or circle around your final answer. If you do not clearly indicate the question numbers, I will take off points. Write as much work as you need to demonstrate to me that you understand the concepts involved. If you have any questions, raise your hand and I will come over to you.

An 800-gallon tank initially contains 300 gallons of brine containing 75 lbs of dissolved salt. Brine containing 6 lbs of salt per gallon flows into the tank at a rate of 3 gallons per minute. The well-stirred mixture then flows out of the tank at a rate of 2 gallons per minute.

Answer the following questions about this scenario:

1. [1 pt] How much salt in lbs is entering the tank per minute?

   Solution: We have

$$\left(\frac{6 \text{ lbs}}{1 \text{ gal}}\right)\left(\frac{3 \text{ gal}}{1 \text{ min}}\right) = \boxed{18 \text{ lbs/min}}$$

2. [2 pts] How much salt in lbs is leaving the tank per minute?

   Solution: Let $A(t)$ be the amount of salt in lbs in the tank at time t minutes. If the brine is well-stirred, then it means that any gallon in the tank contains as much salt as any other gallon. Because the total number of gallons in the tank is changing over time, we need to take that into account. There are 3 gallons entering the tank each minute and 2 gallons leaving it. This gives a net change of 1 gallon added to the tank each minute. Hence, the function that gives the total number of gallons in the tank is $300 + t$. Thus, the amount of salt leaving the tank is

$$\left(\frac{A(t) \text{ lbs}}{300 + t \text{ gal}}\right)\left(\frac{2 \text{ gal}}{1 \text{ min}}\right) = \boxed{\frac{2A(t)}{300 + t} \text{ lbs/min}}$$

3. [2 pts] Set up a differential equation describing the rate of change of salt in lbs in the tank at time t minutes.

   Solution: The rate of change of the salt is given by

$$\frac{dA}{dt} = [\text{Rate in}] - [\text{Rate out}] = \boxed{18 - \frac{2A(t)}{300 + t}}.$$

4. [3 pts] Find the particular solution to # 3.

   Solution: This is a FOLDE, although it isn't quite in the right form. But we quickly rewrite it to get

$$\frac{dA}{dt} + \frac{2A}{300 + t} = 18.$$

Step 1: Find P, Q

$P(t) = \dfrac{2}{300+t}, \quad Q(t) = 18$

Step 2: Find the integrating factor

$$\begin{aligned} u(t) &= e^{\int P(t)\,dt} \\ &= e^{\int \frac{2}{300+t}\,dt} \\ &= e^{2\ln|300+t|} \\ &= e^{\ln|300+t|^2} \\ &= |300+t|^2 \\ &= (300+t)^2 \end{aligned}$$

Here, we drop the absolute values because we may assume the amount of brine in the tank is non-negative.

Step 3: Set up the solution

$$\begin{aligned} A(t)\cdot u(t) &= \int Q(t)u(t)\,dt \\ \Rightarrow \quad A(t)(300+t)^2 &= \int 18(300+t)^2\,dt \\ \Rightarrow \quad A(t)(300+t)^2 &= \frac{18}{3}(300+t)^3 + C \\ \Rightarrow \quad A(t) &= 6(300+t) + \frac{C}{(300+t)^2} \end{aligned}$$

Now, we need to solve for C. We are told that $A(0) = 75$, hence

$$\begin{aligned} 75 &= 6(300+0) + \frac{C}{(300+0)^2} \\ \Rightarrow \quad 75 &= 1800 + \frac{C}{(300)^2} \\ \Rightarrow \quad -1725 &= \frac{C}{(300)^2} \\ \Rightarrow \quad -1725(300)^2 &= C \end{aligned}$$

Thus,

$$A(t) = \boxed{6(300+t) - \frac{1725(300)^2}{(300+t)^2}}.$$

5. [2 pts] How much salt is in the tank after 5 minutes? Round your answer to 2 decimal places.

$$\begin{aligned} A(5) &= 6(300+5) - \frac{1725(300)^2}{(300+5)^2} \\ &= 6(305) - \frac{1725(300)^2}{(305)^2} \\ &\approx \boxed{161.09 \text{ lbs}} \end{aligned}$$